ANSWER TO THE MAKE-UP MIDTERM EXAMINATION

SOLUTION TO THE 1ST QUESTION

(a). A PDE problem is called to be well-posed if it has the following three properties that:

1. Existence: The problem has a solution;
2. Uniqueness: There is at most one solution;
3. Stability: Solution depends continuously on the data given in the problem.

(b). Since $1 - 4 \times 1 \times (-2) = 9 > 0$, the equation is of hyperbolic type.

(c). Let

$$\frac{dx}{dt} = 4,$$

then along the characteristic curve $x(t) = 4t + a$,where a is a fixed constant, the partial differential equation becomes

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = 2u,$$

therefore

$$u(t, x(t)) = e^{2t}a^2,$$

then

$$u(t, x) = e^{2t}(x - 4t)^2.$$

(d). Let

$$\frac{dy}{dx} = -2,$$

then along the characteristic curve $y(x) = -2x + a$,where a is a fixed constant, the partial differential equation becomes

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\frac{dy}{dx} = -2u + 1,$$

therefore

$$u(x, y(x)) = \frac{1}{2}(1 - e^{-2x}) + e^{-2x}u(0, a),$$

which implies for arbitrary $f \in C^1(\mathbb{R})$,

$$u(x, y) = \frac{1}{2}(1 - e^{-2x}) + e^{-2x}f(y + 2x),$$

is a solution to the equation.

SOLUTION TO THE 2ND QUESTION

(a).

Theorem. *Let $u \in C^2(\Omega)$ be harmonic in Ω. Then u satisfies the mean-value property in Ω, i.e.*

$$u(x) = \frac{1}{4\pi r^2} \int_{\partial B_r(x)} u(y) dS_y,$$

or

$$u(x) = \frac{3}{4\pi r^3} \int_{B_r(x)} u(y) dy.$$

Proof. For arbitrary $B_\rho(x) \subset \Omega$, denote $n(x)$ to be the outward normal vector at $x \in \partial B_\rho(x)$, then we have

$$\begin{aligned}\int_{B_\rho(x)} \Delta u(y) dy =& \int_{\partial B_\rho(x)} \nabla u(y) \cdot n(y) dS_y \\ =& \rho^n \int_{|w|=1} \nabla u(x+\rho w) \cdot w dw \\ =& \rho^n \int_{|w|=1} \frac{\partial u(x+\rho w)}{\partial \rho} dw \\ =& \rho^n \frac{\partial}{\partial \rho} \int_{|w|=1} u(x+\rho w) dw,\end{aligned}$$

which implies

$$\frac{\partial}{\partial \rho} \int_{|w|=1} u(x+\rho w) dw = 0,$$

integrating the above inequality from 0 to r, we have

$$\int_{|w|=1} u(x) dw = \int_{|w|=1} u(x+rw) dw,$$

therefore

$$u(x) = \frac{1}{4\pi r^2} \int_{\partial B_r(x)} u(y) dS_y.$$

Moreover, since

$$\int_{B_r(x)} u(y) dS_y = \int_0^r \int_{\partial B_\rho(x)} u(y) dS_y d\rho,$$

we have

$$u(x) = \frac{3}{4\pi r^3} \int_{B_r(x)} u(y) dy.$$

□

(b). Denote $M = \max_{\bar{\Omega}} u(x)$, and define $\Omega_M = \{x \in \Omega : v(x) = M\}$. Then since for arbitrary $x \in \Omega_M$,

$$u(x) = \frac{3}{4\pi r^3} \int_{B_r(x)} u(y) dy, \quad \forall B_r(x) \subset \Omega,$$

which implies x is a interior point of Ω_M, therefore Ω_M is open, since u is continous, Ω_M is also relatively closed in Ω. Suppose u is not constant and it attains its maximum value only in Ω, then Ω_M is not empty, therefore $\Omega_M = \Omega$ which means u is constant, a contradiction! Therefore

$$\max_{\bar{\Omega}} u(x) = \max_{\partial\Omega} u(x).$$

SOLUTION TO THE 3RD QUESTION

(a). By direct computation,

$$\partial_x \left(\frac{\partial_x h}{\sqrt{1+|\partial_x h|^2}} \right) = -\frac{|\partial_x h|^2 \cdot \partial_x^2 h}{(1+|\partial_x h|^2)^{\frac{3}{2}}} + \frac{\partial_x^2 h}{\sqrt{1+|\partial_x h|^2}}$$
$$= \frac{\partial_x^2 h}{(1+|\partial_x h|^2)^{\frac{3}{2}}}.$$

From the equation for h, we have

$$\partial_x \partial_t h - \partial_x \left(\frac{\partial_x^2 h}{1+|\partial_x h|^2} \right) = 0,$$

multiplying the above equation by $\frac{\partial_x h}{\sqrt{1+|\partial h|^2}}$ and integrating the resultant over $[0, 2\pi]$, then

$$\frac{d}{dt} \int_0^{2\pi} \sqrt{1+|\partial_x h|^2} dx + \int_0^{2\pi} \frac{|\partial_x^2 h|^2}{(1+|\partial_x h|^2)^{\frac{5}{2}}} dx = 0,$$

which implies

$$\frac{d}{dt} \int_0^{2\pi} \sqrt{1+|\partial_x h|^2} dx \leq 0.$$

(b). From the equation for h, we have

$$\partial_x \partial_t h - \partial_x \left(\frac{\partial_x^2 h}{1+|\partial_x h|^2} \right) = 0,$$

multiplying the above equation by $\partial_x h$ and integrating the resultant over $[0, 2\pi]$, then

$$\frac{d}{dt} \int_0^{2\pi} |\partial_x h|^2 dx + \int_0^{2\pi} \frac{|\partial_x^2 h|^2}{1+|\partial_x h|^2} dx = 0,$$

which implies

$$\frac{d}{dt} \int_0^{2\pi} |\partial_x h|^2 dx \leq 0.$$

(c).

(i). Since

$$\partial_x(\arctan(\partial_x h)) = -\frac{\partial_x^2 h}{1+|\partial_x h|^2},$$

therefore h satisfies

$$\partial_t h - \partial_x(\arctan(\partial_x h)) = 0,$$

multiplying the above equation by h and integrating the resultant over $[0, 2\pi]$, we have

$$\frac{d}{dt} \int_0^{2\pi} h^2 dx + \int_0^{2\pi} \partial_x h \cdot \arctan(\partial_x h) dx = 0,$$

since $x\arctan(x) \geq 0$, the above equality implies

$$\frac{d}{dt}\int_0^{2\pi} h^2 dx \leq 0.$$

(ii). By direct computation,

$$\begin{aligned}
&\partial_t(\partial_x h \arctan(\partial_x h)) \\
=&\partial_t\partial_x h \cdot \arctan(\partial_x h) + \partial_x h \cdot \partial_t(\arctan(\partial_x h)) \\
=&\left(\arctan(\partial_x h) + \frac{\partial_x h}{1+|\partial_x h|^2}\right)\partial_x\left(\frac{\partial_x^2 h}{1+|\partial_x h|^2}\right).
\end{aligned}$$

Since

$$\frac{d}{dt}\int_0^{2\pi} h^2 dx + \int_0^{2\pi} \partial_x h \arctan(\partial_x h) dx = 0,$$

then

$$\frac{d^2}{dt^2}\int_0^{2\pi} h^2 dx - \int_0^{2\pi} |\partial(\arctan(\partial_x h))|^2 dx = 0,$$

which implies

$$\frac{d^2}{dt^2}\int_0^{2\pi} h^2 dx \geq 0.$$

(iii). By direct computation,

$$\begin{aligned}
\partial_t^2 h =&\partial_t\left(\frac{\partial_x^2 h}{1+|\partial_x h|^2}\right) \\
=&-\frac{\partial_x^2 h \cdot \partial_x h \cdot \partial_t\partial_x h}{(1+|\partial_x h|^2)^2} + \frac{\partial_t\partial_x^2 h}{1+|\partial_x h|^2} \\
=&\partial_x\left(\frac{\partial_t\partial_x h}{1+|\partial_x h|^2}\right).
\end{aligned}$$

(iv). From the equation for h, we have

$$\partial_t^2 h - \partial_t\left(\frac{\partial_x^2 h}{1+|\partial_x h|^2}\right) = 0,$$

multiplying the equation by $\partial_t h$ and integrating the resultant over $[0, 2\pi]$, we have

$$\frac{d}{dt}\int_0^{2\pi} |\partial_t h|^2 dx + \int_0^{2\pi} \frac{|\partial_t\partial_x h|^2}{1+|\partial_x h|^2} dx = 0,$$

which implies

$$\frac{d}{dt}\int_0^{2\pi} |\partial_t h|^2 dx \leq 0.$$

Multiplying the equation for h by $\partial_t h$ and integrating the resultant over $[0, 2\pi]$, we have

$$\int_0^{2\pi} |\partial_t h|^2 dx - \int_0^{2\pi} \left|\frac{\partial_x^2 h}{1+|\partial_x h|^2}\right|^2 dx = 0,$$

then

$$\frac{d}{dt}\int_0^{2\pi} \left|\frac{\partial_x^2 h}{1+|\partial_x h|^2}\right|^2 dx = \frac{d}{dt}\int_0^{2\pi} |\partial_t h|^2 dx \leq 0.$$